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Author(s)	SUZUKI, TAKASHI; ZHANG, XIAO
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Trudinger-Moser inequality for point vortex mean field limit with multi-intensities

TAKASHI SUZUKI AND XIAO ZHANG

We study a variational functional associated with point vortex mean field equation, particularly the extremal case, that is, boundedness of the functional and existence of minimizer.

1 Introduction

In 1949, Onsager [13] used statistical mechanics to describe an ordered structure observed in fluid motion. In the theory of Gibbs, first, the Hamilton system

$$\frac{dq_i}{dt}=\frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt}=-\frac{\partial H}{\partial q_i}, \qquad 1\leq i\leq N,$$

is introduced in the phase space $x=(q_1,\ldots,q_N,p_1,\ldots,p_N)\in \mathbf{R}^{6N}$. It induces the micro-canonical measure

$$d\mu^{H,N}=\frac{1}{\Omega(H)}\cdot\frac{d\Sigma(H)}{|\nabla H|}$$

where $d\Sigma(H)$ and $\Omega(H)$ denote the measure on each level set $\{x\in \mathbf{R}^{6N}\mid H(x)=H\}$ and the weight factor defined by

$$dx=dH\cdot\frac{d\Sigma(H)}{|\nabla H|}$$

and

$$\Omega(H)=\int_{\{H(x)=H\}}\frac{d\Sigma(H)}{|\nabla H|},$$

respectively. Then the thermodynamical relation gives the inverse temperature $\beta=1/(k_BT)$ by

$$\beta=\frac{\partial}{\partial H}\log\Omega(H)$$

from which emerges the canonical measure

$$d\mu^{\beta,N} = \frac{e^{-\beta H}dx}{Z(\beta,N)}, \quad Z(\beta,N) = \int_{\mathbf{R}^{6N}} e^{-\beta H}dx$$

where k_B denotes the Boltzmann constant. Then the mean field limt of the factorized density of $d\mu^{\beta,N}$, that is, the one point pdf, arises as $N \uparrow +\infty$ under the principle of equal a priori probabilities.

Onsager [13] used the vorticity equation of Kirchhoff which is derived from the Euler equation

$$\begin{aligned} &v_t + (v\cdot\nabla)v = -\nabla p, \quad \nabla\cdot v = 0, && \text{in } \Omega\times(0,T) \\ &\nu\cdot v = 0 && \text{on } \partial\Omega\times(0,T), \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbf{R}^2$ is a simply-connected domain with smooth boundary $\partial\Omega$ and ν denotes the outer unit normal vector. If $\omega = \nabla \times v$ is so concentrated as

$$\omega_N(dx,t) = \sum_{i=1}^{N} \alpha_i \delta_{x_i(t)}(dx),$$

equation (1.1) is reduced to

$$\frac{dx_i}{dt} = \nabla_i^\perp \hat{H}_N, \quad 1 \le i \le N$$

for

$$\hat{H}_N(x_1,\ldots,x_N) = \sum_i \frac{\alpha_i^2}{2} R(x_j) + \sum_{i<j} \alpha_i\alpha_j G(x_i,x_j),$$

where

$$\nabla_i^\perp = \begin{pmatrix} \frac{\partial}{\partial x_{i2}} \\ -\frac{\partial}{\partial x_{i1}} \end{pmatrix}, \quad x_i = (x_{i1}, x_{i2}),$$

$G = G(x,x')$ denotes the Green's function, and

$$R(x) = \left[G(x,x') + \frac{1}{2\pi}\log|x-x'| \right]_{x'=x}$$

is the Robin function. For the single intensity case $\alpha_i = \hat{\alpha}$, the equation to which mean field limit of the canonical measure is subject is derived by [5, 12]. Namely, it arises in the high energy limit, $N \uparrow +\infty$ with

$$\hat{\alpha}N = 1, \ \hat{H}_N = H, \ \hat{\alpha}^2 N\hat{\beta} = \beta,$$

and the one-point pdf takes the limit satisfying

$$\rho = \frac{e^{-\beta\psi}}{\int_\Omega e^{-\beta\psi}}, \quad \psi = \int_\Omega G(\cdot, x')\rho(x')dx'. \tag{1.2}$$

The rigorous proof [2, 6] for this limit process is valid if $\lambda = -\beta < 8\pi$ because of the uniqueness of the solution to (1.2) proven by [18]. Equation (1.2) takes the form of the Boltzmann-Poisson equation

$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v} \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

If the distribution of the vortices of the intensity $\alpha\hat{\alpha}$, $\alpha \in [-1,1]$, $N\hat{\alpha} = 1$, is subject to the Borel probability measure $P(d\alpha)$, then (1.3) is replaced by

$$-\Delta v = \lambda \int_{[-1,1]} \frac{\alpha e^{\alpha v}}{\int_\Omega e^{\alpha v}} P(d\alpha) \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega. \tag{1.4}$$

It is the point vortex mean field equation for the case of multi-intensities. A formal derivation of this deterministic distribution is done in [16], but Onsager himself has left a note where (1.4) is shown for the discrete case

$$P(d\alpha) = \sum_{i=1}^{\ell} n^i \delta_{\alpha_i} \tag{1.5}$$

(see [3]).

A different model derived by [8] is the stochastic case where relative intensity $\alpha \in [-1,1]$ is a random variable subject to the distribution function $P(d\alpha)$. Then it follows that

$$-\Delta v = \lambda \frac{\int_{[-1,1]} \alpha e^{\alpha v} P(d\alpha)}{\int_{]-1,1]} \int_\Omega e^{\alpha v} P(d\alpha)}, \quad v|_{\partial\Omega} = 0. \tag{1.6}$$

If the intensities are neutral we have

$$P(d\alpha) = \frac{1}{2}(\delta_1 + \delta_{-1}) \tag{1.7}$$

and then equations (1.4) and (1.6) read

$$-\Delta v = \frac{\lambda}{2} \left(\frac{e^v}{\int_\Omega e^v dx} - \frac{e^{-v}}{\int_\Omega e^{-v} dx} \right), \quad v|_{\partial\Omega} = 0$$

and

$$-\Delta v = \frac{\lambda(e^v - e^{-v})}{\int_\Omega e^v + e^{-v}\,dx}, \quad v|_{\partial\Omega} = 0,$$

respectively.

Equations (1.4) and (1.6) are the Euler-Lagrange equations of the functionals

$$J_\lambda^d(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \int_{[-1,1]} \left[\log \int_\Omega e^{\alpha v}\right] P(d\alpha)$$

and

$$J_\lambda^s(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log \int_{[-1,1]} \left[\int_\Omega e^{\alpha v}\right] P(d\alpha)$$

defined for $v \in H_0^1(\Omega)$, respectively. Then the extremal values of λ for their boundedness is a fundamental factor to prescribe the critical state of many stationary point vortices. We study these functionals on

$$E = \{v \in H^1(\Omega) \mid \int_\Omega v = 0\}$$

where Ω is a Riemann surface without boundary.

First, it is obvious that

$$J_\lambda^d \geq J_\lambda^s$$

by Jensen's inequality. Next, the Trudinger-Moser-Fontana inequality [4]

$$\int_\Omega e^{4\pi w^2} \leq C, \quad \forall w \in E, \ \|\nabla w\|_2 \leq 1$$

implies

$$\inf_E J_{8\pi}^s > -\infty.$$

In fact, we have

$$\alpha v \leq \frac{1}{16\pi}\|\nabla v\|_2^2 + 4\pi\alpha^2 \cdot \frac{v^2}{\|\nabla v\|_2^2}$$

and hence

$$\log \int_{[-1,1]} \left[\int_\Omega e^{\alpha v}\right] P(d\alpha) \leq \frac{\|\nabla v\|_2^2}{16\pi} + C, \qquad v \in E.$$

In fact the value $\lambda = 8\pi$ is actually the extremal for J^s to be bounded by the following theorem.

Theorem 1 ([15]). *If*

$$\sup supp\ P = +1 \quad or \quad \inf supp\ P = -1 \tag{1.8}$$

it holds that $\inf_E J_\lambda = -\infty$ *for* $\lambda > 8\pi$.

Proof. We assume $\sup \mathrm{supp}\ P = +1$ without loss of generality. If $\alpha > 0$ we have

$$ve^{\alpha v} \geq v, \quad \forall v \in \mathbf{R}$$

and hence

$$\frac{d}{d\alpha}\int_\Omega e^{\alpha v} = \int_\Omega ve^{\alpha v} \geq \int_\Omega v = 0$$

Then it holds that

$$\begin{aligned}
\log \int_{[-1,1]} \left[\int_\Omega e^{\alpha v}\right] P(d\alpha) &\geq \log \int_{[1-\delta,1]} \int_\Omega e^{\alpha v} P(d\alpha) \\
&\geq \log \int_\Omega e^{(1-\delta)v} + \log P[1-\delta, 1]
\end{aligned}$$

for $0 < \delta < 1$ and $v \in E$. Writing $w = (1-\delta)v$, then we obtain

$$\begin{aligned}
J_\lambda^s(v) &\leq \frac{1}{2}\cdot\frac{1}{(1-\delta)^2}\|\nabla w\|_2^2 - \lambda \log \int_\Omega e^w + C_\delta \\
&= \frac{1}{(1-\delta)^2}\left\{\frac{1}{2}\|\nabla w\|_2^2 - \lambda(1-\delta)^2 \log \int_\Omega e^w\right\} + C_\delta.
\end{aligned}$$

Given $\lambda > 8\pi$, we have $0 < \delta \ll 1$ such that $\tilde{\lambda} = \lambda(1-\delta)^2 > 8\pi$. Then it follows that

$$\inf_E J_{\tilde{\lambda}}^s = -\infty$$

from $\inf_E J_{\tilde{\lambda}}^0 = -\infty$, where

$$J_\lambda^0(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log \int_\Omega e^v. \tag{1.9}$$

□

Now we turn to the extremal value for J^d defined by

$$\lambda_* = \sup\{\lambda \mid \inf_E J_\lambda^d > -\infty\}.$$

If $\lambda < \lambda_*$ then J^d_λ takes minimizer on E which solves

$$-\Delta v = \lambda \int_{[-1,1]} \alpha \left[\frac{e^{\alpha v}}{\int_\Omega e^{\alpha v}} - \frac{1}{|\Omega|} \right] P(d\alpha), \quad \int_\Omega v = 0. \tag{1.10}$$

If the minimizer of $J^d_{\lambda_*}$ on E does not exist, there will be a formation of singularity of ground states at the critical level of negative inverse temperature $\lambda = \lambda_*$. Furthermore, the profile of its singularity is associated with the boundedness of the extremal functional indicated by

$$\inf_E J^d_{\lambda_*} > -\infty. \tag{1.11}$$

Thus we are addressed by three questions at this moment; prescribing the exact value λ_*, boundedness of the extremal functional (1.11), and the existence or non-existence of the minimizer of $J^d_{\lambda_*}$ on E. In fact, Ohtsuka-Suzuki [10] showed $\lambda_* = 16\pi$ for the neutral case (1.7).

In 2010, Ohtsuka-Ricciardi-Suzuki [9] prescribed the profile of singular limits of the solution to (1.10), and derived a rough estimate,

$$\lambda_* \geq \inf \left\{ \frac{8\pi}{\int_{[-1,0]} \alpha^2 P(d\alpha)}, \frac{8\pi}{\int_{[0,1]} \alpha^2 P(d\alpha)} \right\}.$$

The exact value of λ_*, however, had been obtained for the discrete case (1.5) by [17], represented in the dual form (see [19]). Taking the limit of this inequality, we obtain the following theorem.

Theorem 2 ([14]). *Under the assumption of (1.8) it holds that*

$$\lambda_* = \inf \left\{ \frac{8\pi P(K_\pm)}{\left[\int_{K_\pm} \alpha P(d\alpha)\right]^2} \middle| K_\pm \subset I_\pm \cap \operatorname{supp} P \right\}, \tag{1.12}$$

where $I_+ = [0,1]$ *and* $I_- = [-1,0]$.

To approach (1.11), here we take $\lambda_k \uparrow \lambda_*$ and the minimizer v_k of $\inf_E J^d_{\lambda_k}$. This $(v, \lambda) = (v_k, \lambda_k)$ is a solution to (1.10) and if $\{v_k\} \subset E$ is compact, then we have (1.11) with a minimizer. If this is not the case we apply [9] to get the following lemma.

Lemma 1. *If the above* $\{v_k\} \subset E$ *is non-compact, then passing to a subsequence we obtain*

$$\frac{\lambda_k e^{\alpha v_k}}{\int_\Omega e^{\alpha v_k}} dx P(d\alpha) \rightharpoonup \mu(dx P(d\alpha)) \quad \text{in } \mathcal{M}(\Omega \times [-1,1])$$

for

$$\mu(dxP(d\alpha)) = \left[\sum_{x_0\in\mathcal{S}} m(x_0,\alpha)\delta_{x_0}(dx) + r(x,\alpha)dx\right] P(d\alpha), \qquad (1.13)$$

where

$$m(x_0,\alpha)\geq 0, \quad 0\leq r = r(x,\alpha)\in L^1(\Omega\times[-1,1], dxP(d\alpha)$$

and $\mathcal{S} = \mathcal{S}_+\cup\mathcal{S}_-$ *with*

$$\mathcal{S}_\pm = \{x_0 \mid \exists x_k\to x_0 \text{ s.t. } v_k(x_k)\to\pm\infty\}$$

with $\sharp\mathcal{S} < +\infty$. *Furthermore, it holds that*

$$8\pi\int_{[-1,1]} m(x_0,\alpha)P(d\alpha) = \left\{\int_{[-1,1]} \alpha m(x_0,\alpha)P(d\alpha)\right\}^2 \qquad (1.14)$$

$$4\pi\leq n_\pm(x_0) = \int_{I_\pm} |\alpha| m(x_0,\alpha)P(d\alpha), \quad \forall x_0\in\mathcal{S}_\pm. \qquad (1.15)$$

Henceforth, we assume the non-compactness of the above $\{v_k\}\subset E$ although the property described in Lemma 1 is valid to any non-compact solution sequence to (1.10). If $r=0$ we say that the *residual vanishing* occurs to (1.13). Then we obtain the following lemma.

Lemma 2 ([20]). *Let* $P(d\alpha)$ *be non-atomic, supp* $P\subset I_+$,

$$\sup\{\alpha\in I_+ \mid P([0,\alpha)) = 0\} > \frac{1}{2}\int_{I_+}\alpha P(d\alpha),$$

and

$$\frac{1}{(\int_{I_+}\alpha P(d\alpha))^2} < \frac{P(K_+)}{(\int_{K_+}\alpha P(d\alpha))^2}$$

for any $K_+\subset I_+\cap$ *supp* P *satisfying* $K_+\neq I_+$, $P(K_+)<1$. *Then it follows that (1.11) under the assumption of the residual vanishing of* $\{v_k\}\subset E$ *defined above.*

The propery (1.11) is valid for the discrete case (1.5) (see [17]). Hence there may be the other approach of evaluating its bound uniformly. Here we note that this approach was successful for the sub-critical case [14], and also that any counter example to (1.11) has not yet be known. Thus there may be a chance for (1.11) to be proven by a limit process similar to the sub-critical case. Actually we expect (1.11) for all cases.

In contrast, the argument taken by this paper may have an advantage of picking up the case of the existence of minimizers. More precisely, if we get a contradiction from the non-compactness of the above $\{v_k\} \subset E$, then there must be a minimizer to $J^s_{\lambda_*}$ on E. So far, the argument employed here guarantees (1.11) for both *clustered* and *separated* cases of $P(d\alpha)$. We have, furthermore, the existence of minimizer in the latter case. This property arises even under slight perturbations of $J^0_{8\pi}$ defined by (1.9), which may be surprising because $J^0_{8\pi}$ itself does not always take any minimizers on E.

This paper is composed of three sections. In §2 we study the residual vanishing and related properties. Then the notion of *partially compact* is introduced and studied in § 3.

2 Residual Vanishing

The proof of the following fact may be useful to observe the role of residual vanishing for (1.11) to be valid.

Proposition 1. *Let $P(d\alpha) = \delta_1$ and define the sequence $\{v_k\} \subset E$ as in the previous section with $\lambda_k \uparrow \lambda_*$. Then it holds that*

$$J^d_{\lambda_k}(v_k) = O(1).$$

Proof. We have $\lambda_* = 8\pi$ and

$$-\Delta v_k = \lambda_k \left(\frac{e^{v_k}}{\int_\Omega e^{v_k}} - \frac{1}{|\Omega|} \right), \quad \int_\Omega v_k = 0.$$

Since $\sharp S = 1$ we may assume

$$v_k(0) \to +\infty, \qquad \int_\Omega e^{v_k} \to +\infty.$$

Then $\xi_k = v_k - \log \int_\Omega e^{v_k}$ satisfies

$$\int_\Omega e^{\xi_k} = 1, \quad e^{\xi_k} \rightharpoonup \delta_0.$$

Y.Y. Li's estimate [7] now guarantees

$$\left|\xi_k(X) - \log \frac{e^{\xi_k(0)}}{\left(1 + \frac{\lambda_k}{8} e^{\xi_k(0)} |X|^2\right)^2}\right| + \left|\xi_k(0) + \overline{\xi_k}\right| \leq C$$

for $|X| \ll 1$, where X denotes the iso-thermal chart and

$$\overline{\xi_k} = \frac{1}{|\Omega|} \int_\Omega \xi_k.$$

Here we have $\log \int_\Omega e^{v_k} = -\overline{\xi_k}$ and also

$$\begin{aligned}
\|\nabla v_k\|_2^2 &= \langle -\Delta v_k, v_k\rangle = -\lambda_k (e^{\xi_k} - \frac{1}{|\Omega|}, v_k) = \lambda_k \int_\Omega e^{\xi_k} v_k \\
&= \lambda_k \left(\int_\Omega e^{\xi_k} \xi_k + \log \int_\Omega e^{v_k} \right) = \lambda_k \left(\int_\Omega e^{\xi_k} \xi_k - \overline{\xi_k} \right),
\end{aligned}$$

which implies

$$\begin{aligned}
\frac{2}{\lambda_k} J_{\lambda_k}(v_k) &= \int_\Omega \xi_k e^{\xi_k} + \overline{\xi_k} \\
&= \int_\Omega (\xi_k - \xi_k(0)) e^{\xi_k} + \left(\xi_k(0) + \overline{\xi_k}\right) = O(1).
\end{aligned}$$

□

The next observation is the following lemma. It shows what is emerged from the residual vanishing if $P(d\alpha)$ is one-sided.

Lemma 3. *Assume supp $P \subset I_+$ and the residual vanishing for $\{v_k\} \subset E$ defined in the previous section. Then it holds that $\sharp S = 1$ and (1.12) is attained by $K_+ = I_+$, that is,*

$$\lambda_* = \frac{8\pi}{\left(\int_{I_+} \alpha P(d\alpha)\right)^2} \tag{2.1}$$

Proof. By (1.13) with $r = 0$ we have

$$\lambda_* = \sum_{x_0 \in S} m(x_0, \alpha), \quad P\text{-a.e. } \alpha,$$

while the first equality of (1.14) reads

$$8\pi \int_{I_+} m(x_0, \alpha) P(d\alpha) = \left\{ \int_{I_+} \alpha m(x_0, \alpha) P(d\alpha) \right\}^2, \quad \forall x_0 \in \mathcal{S} \tag{2.2}$$

by supp $P \subset I_+$. Then it holds that

$$\begin{aligned} 8\pi\lambda_* &= \sum_{x_0 \in \mathcal{S}} \left\{ \int_{I_+} \alpha m(x_0, \alpha) P(d\alpha) \right\} \\ &\leq \left\{ \int_{I_+} \sum_{x_0 \in \mathcal{S}} \alpha m(x_0, \alpha) P(d\alpha) \right\}^2 = \left\{ \int_{I_+} \alpha \lambda_* P(d\alpha) \right\}^2 \end{aligned} \tag{2.3}$$

and hence

$$\lambda_* \geq \frac{8\pi}{\left\{ \int_{I_+} \alpha P(d\alpha) \right\}^2}.$$

Therefore, (1.12) is attained for $K_+ = I_+$ and the equality is valid in (2.3) which means $\sharp \mathcal{S} = 1$. □

The following lemma is useful to ensure the residual vanishing.

Lemma 4. *Given a relatively open set denoted by $I_0 \subset I$, we have*

$$r = 0, \quad dxP(d\alpha)\text{-a.e. on } \Omega \times I_0 \tag{2.4}$$

if and only if any $k \to \infty$ admits $\{k'\} \subset \{k\}$ such that

$$\int_\Omega e^{\alpha v'_k} \to +\infty, \quad P\text{-a.e. } \alpha \in I_0. \tag{2.5}$$

Proof. First, assume (2.5), and take $\psi \in C(\Omega \setminus \mathcal{S})$. Then it holds that

$$\left\langle \psi, \frac{e^{\alpha v'_k}}{\int_\Omega e^{\alpha v'_k}} \right\rangle \to 0, \quad P\text{-a.e. } \alpha \in I_0.$$

Here we have

$$\frac{1}{|\Omega|} \int_\Omega e^{\alpha v'_k} \geq \exp\left(\frac{1}{|\Omega|} \int_\Omega \alpha v'_k \right) = 1 \tag{2.6}$$

and hence

$$\int_I \varphi \left\langle \psi, \frac{e^{\alpha v'_k}}{\int_\Omega e^{\alpha v'_k}} \right\rangle P(d\alpha) \to 0 \tag{2.7}$$

by the dominated convergence theorem, where $\varphi \in C_0(I_0)$ is arbitrary. Then it follows that

$$\int_I \varphi \langle \psi, r(\cdot, \alpha) \rangle \, P(d\alpha) = 0$$

from (1.13), which implies (2.4).

If (2.4) is the case, conversely, it holds that

$$\int_I \varphi \left\langle \psi, \frac{e^{\alpha v_k}}{\int_\Omega e^{\alpha v_k}} \right\rangle P(d\alpha) \to 0$$

for any $0 \leq \psi \in C(\Omega \setminus \mathcal{S})$ and $0 \leq \varphi \in C_0(I_0)$. Passing to a sub-sequence, we obtain

$$\left\langle \psi, \frac{e^{\alpha v_k}}{\int_\Omega e^{\alpha v_k}} \right\rangle \to 0, \quad P\text{-a.e. } \alpha \in I_0$$

by a diagonal argument. Here the elliptic regularity to (1.10) combined with (2.6) guarantees $\|v_k\|_{L^\infty(\omega)} = O(1)$, where $\omega \subset \Omega \setminus \mathcal{S}$ is an open set. Hence

$$\int_\Omega e^{\alpha v_k} \to +\infty, \quad P\text{-a.e. } \alpha \in I_0$$

for this subsequence and the proof is complete. $\square$

Now we show the following theorem.

Theorem 3. *If supp $P \subset I_+$ then $r(\cdot, \alpha) = 0$ a.e. in Ω for $\alpha > 1/2$. In particular, the residual vanishing occurs to $\{v_k\} \subset E$ defined in the previous section, provided that supp $P \subset (1/2, 1]$.*

Proof. We shall show

$$\int_\Omega e^{\alpha v_k} \to +\infty, \quad \forall \alpha > 1/2. \tag{2.8}$$

In fact, (2.2) implies

$$\int_{I_+} \alpha m(x_0, \alpha) P(d\alpha) \geq 8\pi, \quad \forall x_0 \in \mathcal{S} \tag{2.9}$$

and the right-hand side of (1.10) for $(\lambda, v) = (\lambda_k, v_k)$ takes the limit

$$\int_{I_+} \alpha \mu(dx P(d\alpha)) - \frac{\lambda_*}{|\Omega|} \int_{I_+} \alpha P(d\alpha)$$

in $\mathcal{M}(\Omega)$. Here we have

$$\int_{I_+} \alpha\mu(dxP(d\alpha)) \geq \sum_{x_0\in\mathcal{S}} \int_{I_+} \alpha m(x_0,\alpha)\delta_{x_0}(dx) \geq 8\pi \sum_{x_0\in\mathcal{S}} \delta_{x_0}(dx)$$

by (2.9). Since $\sharp\mathcal{S} \neq \emptyset$, equation (1.10) implies (2.8) by an argument used in [1]. $\square$

We conclude this section with the following examples. Henceforth, $v_k \in E$ denotes the minimizer of $J^s_{\lambda_k}$ such that $\lambda_k \uparrow \lambda_*$.

Example 1. $P = \frac{1}{2}(\delta_1 + \delta_\gamma)$, $0 < \gamma < 1$.

Since

$$\frac{8\pi P(K_+)}{\left\{\int_{K_+} \alpha P(d\alpha)\right\}^2} = \begin{cases} \frac{32\pi}{(1+\gamma)^2}, & K_+ = \{1,\gamma\} \\ 16\pi, & K_+ = \{1\} \\ \frac{16\pi}{\gamma^2}, & K_+ = \{\gamma\} \end{cases}$$

it holds that

$$\lambda_* = \inf\{16\pi, \frac{16\pi}{\gamma^2}, \frac{32\pi}{(1+\gamma)^2}\} = \begin{cases} 16\pi, & \gamma < \sqrt{2}-1 \\ \frac{32\pi}{(1+\gamma)^2}, & \gamma \geq \sqrt{2}-1. \end{cases} \tag{2.10}$$

Therefore, the residual vanishing does not occur for $\gamma < \sqrt{2}-1$ by Lemma 3. On the contrary, we have the residual vanishing if $\gamma > 1/2$ by Theorem 3. Next, (1.13) implies

$$\lambda_* \int_{I_+} \varphi P(d\alpha) = \int_{I_+} \left[\int_\Omega r(x,\alpha)dx + \sum_{x_0\in\mathcal{S}} m(x_0,\alpha)\right] \varphi(\alpha)P(d\alpha) \tag{2.11}$$

for any $\varphi \in C(I_+)$. Regarding supp $P = \{1,\gamma\}$, we put $m_\alpha(x_0) = m(x_0,\alpha)$ for $\alpha = 1,\gamma$. Then we obtain

$$\lambda_* \geq \sum_{x_0\in\mathcal{S}} m_1(x_0),\ \sum_{x_0\in\mathcal{S}} m_\gamma(x_0). \tag{2.12}$$

Equality (1.14), on the other hand, is reduced to (2.2), which means

$$16\pi(m_1(x_0) + m_\gamma(x_0)) = (m_1(x_0) + \gamma m_\gamma(x_0))^2, \quad \forall x_0 \in \mathcal{S}. \tag{2.13}$$

By (2.12)-(2.13) we can conclude $\sharp\mathcal{S} = 1$. We put $m_\alpha = m_\alpha(x_0)$ for $x_0 \in \mathcal{S}$, $\alpha = 1,\gamma$. If $\gamma > \sqrt{2}-1$, then $\lambda_* = \frac{32\pi}{(1+\gamma)^2} < 16\pi$. There is only one pair of (m_γ, m_1) with $m_\gamma, m_1 > 0$, satisfying (2.12) and (2.13), that is,

$$m_1 = m_\gamma = \frac{32\pi}{(1+\gamma)^2}. \tag{2.14}$$

Then equalities arise in both inequalities in (2.12), which implies $r(\cdot, \alpha) = 0$ a.e. for $\alpha = 1, \gamma$. Thus we obtain the residual vanishing.

If $\gamma \leq \sqrt{2} - 1$ then we have $\lambda_* = 16\pi$. If $\gamma = \sqrt{2} - 1$, there arise the cases of $(m_\gamma, m_1) = (16\pi, 16\pi)$ and $(m_\gamma, m_1) = (0, 16\pi)$ from (2.12)-(2.13). In the former case we have the residual vanishing, while in the latter case we do not have $r_\gamma \equiv r(\cdot, \gamma) = 0$ a.e. any more. We may call it *mass separation*, regarding $m_\gamma = 0$. If $\gamma < \sqrt{2} - 1$, only $(m_\gamma, m_1) = (0, 16\pi)$ satisfies (2.12)-(2.13). Hence we always have non-residual vanishing and mass separation.

Assuming $m_\gamma = 0$, we take $0 < R \ll 1$ such that

$$\int_{\mathcal{S}_R} r_\gamma < 4\pi, \quad \mathcal{S}_R = \bigcup_{x_0 \in \mathcal{S}} B(x_0, R)$$

and define $v_k^\gamma = v_k^\gamma(x)$ by

$$-\Delta v_k^\gamma = \frac{\lambda_k}{2}\left(\frac{e^{\gamma v_k}}{\int_\Omega e^{\gamma v_k}} - \frac{1}{|\Omega|}\right), \quad \int_\Omega v_k^\gamma = 0.$$

Then it holds that $\|v_k^\gamma\|_\infty \leq C$ by Brezis-Merle's inequality [1] and Lemma 1. Now, $v_k^1 = v_k - v_k^\gamma$ satisfies

$$-\Delta v_k^1 = \frac{\lambda_k}{2}\left(\frac{V_k e^{v_k^1}}{\int_\Omega V_k e^{v_k^1}} - \frac{1}{|\Omega|}\right), \quad \int_\Omega v_k^1 = 0$$

for $V_k = e^{v_k^\gamma} > 0$. We have readily shown that $\{v_k^\gamma\}$ is compact in $C^{2,\alpha}(\Omega)$, $0 < \alpha < 1$, and $\lambda_k \uparrow 16\pi$ with $\|v_k^1\|_\infty \uparrow +\infty$. In particular, it holds that $J_{\lambda_k}^d(v_k) = \hat{J}_k(v_k^1) + O(1)$ for

$$\hat{J}_k(v) = \frac{1}{2}\|\nabla v\|_2^2 - \frac{\lambda_k}{2}\log\int_\Omega V_k e^v.$$

Here Y.Y. Li's estimate is available even for this case of variable coefficients, which implies $\hat{J}_k(v_k^1) = O(1)$ similarly to Proposition 1. Hence it holds that (1.11).

Summing up, if $\gamma > \sqrt{2} - 1$ then we have the residual vanishing where a refined version of Lemma 2 is expected to apply to guarantee (1.11). If $\gamma < \sqrt{2} - 1$ there arises mass separtion, and then (1.11) by a modification of Proposition 1. Although the case $\gamma = \sqrt{2} - 1$ has not yet been settled down, the above study may suggest the following. First, if $P(d\alpha)$ is sufficiently separated aroud $\alpha = 1$ and $\alpha = 0$, mass separation and consequently non-residual vanishig occur. Then the property (1.11) is reduced to the case that

$P(d\alpha)$ is clustered near $\alpha = 1$. Second, if $P(d\alpha)$ is clustered near $\alpha = 1$ then the residual vanishing occurs, which will make Lemma 2 available. Actually, the proof of Lemma 2 is based on a kind of Y.Y. Li's estimate.

We have to note, however, that the weight of two delta functions are fixed here. Actually, if the positions of two delta functions are sufficiently clustered and their weights are concentrated at $\alpha = 1$, then we have a different phenomena, which guarantees that $J^d_{\lambda_*}$ is attained.

Example 2. $P = \tau\delta_1 + (1-\tau)\delta_\gamma$, $0 < \gamma < 1$, $0 < \tau < 1$.

We shall follow the notations used in the previous example. First observation is that

$$\frac{8\pi P(K_+)}{\left\{\int_{K_+} \alpha P(d\alpha)\right\}^2} = \begin{cases} \frac{8\pi}{(\tau+(1-\tau)\gamma)^2}, & K_+ = \{1,\gamma\} \\ \frac{8\pi}{\tau}, & K_+ = \{1\} \\ \frac{8\pi}{(1-\tau)\gamma^2}, & K_+ = \{\gamma\} \end{cases}$$

implies

$$\lambda_* = \begin{cases} \frac{8\pi}{\tau}, & \gamma < \frac{\sqrt{\tau}}{1+\sqrt{\tau}} \\ \frac{8\pi}{(\tau+(1-\tau)\gamma)^2}, & \gamma > \frac{\sqrt{\tau}}{1+\sqrt{\tau}}, \end{cases}$$

except for the critical case $\gamma = \frac{\sqrt{\tau}}{1+\sqrt{\tau}}$. Inequality (2.12) is still valid, while (2.13) is replaced by

$$8\pi(\tau m_1(x_0) + (1-\tau)m_\gamma(x_0)) = (\tau m_1(x_0) + \gamma(1-\tau)m_\gamma(x_0))^2, \quad \forall x_0 \in \mathcal{S}.$$

Then we can confirm $\sharp\mathcal{S} = 1$ similarly.

Treating the separative case $\gamma < \frac{\sqrt{\tau}}{1+\sqrt{\tau}}$, we observe that the line $m_1 = 8\pi/\tau$ in $m_\gamma m_1$ plane crosses the curve

$$8\pi(\tau m_1 + (1-\tau)m_\gamma) = (\tau m_1 + \gamma(1-\tau)m_\gamma)^2 \tag{2.15}$$

at $m_\gamma = 0$ and $m_\gamma = \frac{1-2\gamma}{\gamma^2(1-\tau)} \cdot 8\pi$, recalling that $\gamma < 1/2$ follows from $\gamma < \frac{\sqrt{\tau}}{1+\sqrt{\tau}}$. Since

$$\lambda_* \geq m_1,\ m_\gamma$$

we have mass separation, $m_\gamma = 0$, provided that $\frac{1}{\tau} < \frac{1-2\gamma}{\gamma^2(1-\tau)}$, i.e., $\gamma < -\sigma + \sqrt{\sigma^2+\sigma}$, $\sigma = \frac{\tau}{1-\tau}$. In such a case, (1.11) is reduced to the boundedness of $\tilde{J}_k(v_k)$, where $\tilde{v}_k \in E$, $\|\tilde{v}_k\|_\infty \to +\infty$,

$$\tilde{J}_k(v) = \frac{1}{2}\|\nabla v\|_2^2 - \mu_k \log \int_\Omega V_k e^v. \tag{2.16}$$

$\mu_k \uparrow \tau\lambda_* = 8\pi$, $V_k = e^{v_k^1}$ with $\{v_k^1\}$ compact in $C^{2,\alpha}(\Omega)$, $0 < \alpha < 1$. This property is actually the case by the proof of Proposition 1. Hence (1.11) arises if (γ, τ) is in the above region.

In the clustered case

$$\gamma > \frac{\sqrt{\tau}}{1+\sqrt{\tau}}, \tag{2.17}$$

it holds that $\lambda_* = \frac{8\pi}{(\tau+(1-\tau)\gamma)^2}$. In this case the curve (2.15) in $m_\gamma m_1$ plane crosses the line $m_1 = \lambda_*$ once, with the m_γ-component of the crossing point denoted by m_γ^*. If

$$(1-\tau)m_\gamma^* < 4\pi \tag{2.18}$$

then we apply Brezis-Merle's inequality as in Example 1. We obtain $\mu_k \uparrow \tau\lambda_*$, $\{V_k\}$, $V_k > 0$, compact in $C^{2,\alpha}(\Omega)$, $0 < \alpha < 1$, and $\tilde{v}_k \in E$ satisfying

$$-\Delta\tilde{v}_k = \mu_k \left(\frac{V_k e^{\tilde{v}_k}}{\int_\Omega V_k e^{\tilde{v}_k}} - \frac{1}{|\Omega|} \right), \quad \int_\Omega \tilde{v}_k = 0.$$

Since $\tau\lambda_* < 8\pi$, however, this $\{\tilde{v}_k\} \subset E$ is compact. Therefore, so is true for the sequence $\{v_k\} \subset E$ defined in the previous section. Hence $\inf_E J_{\lambda_*}^d$ is attained.

Finally, we shall show that (2.17) with (2.18) actually arises in the case of $0 < 1-\tau \ll 1$ and $1/2 < \gamma < 1$. First, given $1/2 < \gamma < 1$, we have (2.17) for $0 < 1-\tau \ll 1$. Next, plugging $m_1 = \lambda_*$ into (2.15), we obtain

$$\begin{aligned} &8\pi \left\{ \frac{8\pi\tau}{(\tau+(1-\tau)\gamma)^2} + (1-\tau)m_\gamma^* \right\} \\ &\quad = \left\{ \frac{8\pi\tau}{(\tau+(1-\tau)\gamma)^2} + \gamma(1-\tau)m_\gamma^* \right\}^2, \end{aligned}$$

which implies

$$\begin{aligned} &\frac{64\pi^2\tau}{(\tau+(1-\tau)\gamma)^4} \left\{ -\tau + 2\gamma + (1-\tau)\gamma^2 \right\} \\ &\quad = \frac{16\pi\gamma m_\gamma^*}{(\tau+(1-\tau)\gamma)^2} + \gamma^2(1-\tau)(m_\gamma^*)^2. \end{aligned}$$

Then it follows that

$$\lim_{\tau\uparrow 1} m_\gamma^* = 4\pi \cdot \frac{2\gamma - 1}{\gamma}$$

for each $1/2 < \gamma < 1$ and hence (2.18) for $0 < 1-\tau \ll 1$.

3 Partially Compact

If P is divided into two parts, and one of its total collapse mass is less than 4π then (1.11) is reduced to that of the other part. We call such a case the *partially compact.* It is obvious that mass separation implies both non-residual vanishing and partially compact. This section is devoted to the general criterion for blowup vanishing to occur. We deal with the cases of one-sided and changing-sign $P(d\alpha)$, individually.

The first theorem is just a generalization of Example 2.

Theorem 4. *Let* $P = \tau P_\beta + (1-\tau)P_\gamma$, *where* $0 < \tau < 1$, $0 < \gamma < \beta < 1$, *and*P_β *and* P_γ *are Borel probability measures on* $[0,\gamma]$ *and* $[\beta, 1]$, *respectively. If* $1/2 < \gamma < 1$ *then* $\inf_E J^d_{\lambda_*}$ *is attained, provided that* $0 < 1 - \tau \ll 1$.

Proof. Assume the contrary, and let $\{v_k\} \subset E$ be the non-compact sequence defined in §1. Then it holds that

$$\frac{8\pi}{\left\{\tau \int_{[\beta,1]} \alpha P_\beta(d\alpha) + (1-\tau)\int_{[0,\gamma]} \alpha P_\gamma(d\alpha)\right\}^2} \geq \lambda_* \tag{3.1}$$

$$\lambda_* \geq \sum_{x_0 \in \mathcal{S}} \int_{[\beta,1]} m(x_0, \alpha) P_\beta(d\alpha), \sum_{x_0 \in \mathcal{S}} \int_{[0,\gamma]} m(x_0,\alpha) P_\gamma(d\alpha) \tag{3.2}$$

and

$$\begin{aligned} &8\pi \left\{\tau \int_{[\beta,1]} m(x_0,\alpha) P_\beta(d\alpha) + (1-\tau)\int_{[0,\gamma]} m(x_0,\alpha)P_\gamma(d\alpha)\right\} \\ &= \left\{\tau \int_{[\beta,1]} \alpha m(x_0,\alpha) P_\beta(d\alpha) + (1-\tau)\int_{[0,\gamma]} \alpha m(x_0,\alpha)P_\gamma(d\alpha)\right\}^2 \end{aligned} \tag{3.3}$$

for each $x_0 \in \mathcal{S}$. Fix $x_0 \in \mathcal{S}$, and put

$$X = \int_{[0,\gamma]} m(x_0,\alpha)P_\gamma(d\alpha), \quad Y = \int_{[\beta,1]} m(x_0,\alpha) P_\beta(d\alpha).$$

As we have seen, if $X < 4\pi$ and $\tau\lambda_* < 8\pi$, there is a contradiction, and hence $\inf_E J^d_{\lambda_*}$ is attained.

First, $\tau\lambda_* < 8\pi$ if

$$\frac{\tau}{(\tau + (1-\tau)\gamma)^2} < 1 \tag{3.4}$$

by (3.1). Here, (3.4) means (2.17). Next, (3.2) and (3.3) imply

$$8\pi(\tau Y + (1-\tau)X) \leq (\tau Y + (1-\tau)\gamma X)^2, \quad X, Y \leq \lambda_*.$$

Since $\tau\lambda_* < 8\pi$ is achieved, X is uniquely determined as

$$8\pi(\tau Y + (1-\tau)X) = (\tau Y + (1-\tau)\gamma X)^2, \quad Y = \lambda_*.$$

Hence both $\tau\lambda_* < 8\pi$ and $X < 4\pi$ is achieved if $1/2 < \gamma < 1$ is given and $0 < 1-\tau \ll 1$ as in Example 2. $\square$

The next theorem is concerned with the changing-sign case, where (1.15) is used.

Theorem 5. *If*

$$\inf \left\{ \frac{P(K_\pm)}{\left\{\int_{K_\pm} \alpha P(d\alpha)\right\}^2} \mid K_\pm \subset I_\pm \cap \mathit{supp}\ P \right\} \cdot \int_{I_\pm} |\alpha| P(d\alpha) < c_\pm \tag{3.5}$$

for $c_- = 1$ *and* $c_+ = 1 + \frac{\sqrt{5}}{2}$ *then it holds that* $\mathcal{S}_- = \emptyset$.

Proof. Fix $x_0 \in \mathcal{S}_-$, and put

$$X_\pm = \int_{I_\pm} |\alpha| m(x_0, \alpha) P(d\alpha) \le \int_{I_\pm} m(x_0, \alpha) P(d\alpha) = Y_\pm.$$

First, we have $\lambda_* \ge m(x_0, \alpha)$, P-a.e. α, and therefore,

$$\begin{aligned} X_\pm &\le \lambda_* \int_{I_+} |\alpha| P(d\alpha) \\ &= 8\pi \cdot \inf \left\{ \frac{P(K_\pm)}{\left\{\int_{K_\pm} \alpha P(d\alpha)\right\}^2} \mid K_\pm \subset I_\pm \cap \mathrm{supp}\ P \right\} \\ &\quad \cdot \int_{I_\pm} |\alpha| P(d\alpha). \end{aligned} \tag{3.6}$$

Next, (1.14) implies

$$\begin{aligned} &\left\{ \int_{[-1,1]} \alpha m(x_0, \alpha) P(d\alpha) \right\}^2 = (X_+ - X_-)^2 \\ &= 8\pi \int_{[-1,1]} m(x_0, \alpha) P(d\alpha) = 8\pi(Y_+ + Y_-) \\ &\ge 8\pi(X_+ + X_-)^2. \end{aligned} \tag{3.7}$$

Here we have

$$4\pi \le X_- < 8\pi \tag{3.8}$$

by (1.15), (3.6), and (3.5) with $c_- = 1$. Then (3.7) implies

$$X_+ \ge 4(2+\sqrt{5})\pi$$

(see [11]), which contradicts (3.6) and (3.5) with $c_+ = 4(2+\sqrt{5})\pi$. $\square$

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Division of Mathematical Science
Department of Systems Innovation
Graduate School of Engineering Scinece
Osaka University

Machimaneyamacho 1-3
Toyonakashi 560-8531, Japan